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ESTIMATION AND TESTS-OF-FIT FOR THE THREE PARAMETER WEIBULL DISTRIBUTION

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Summary

Estimation techniques are given for the three-parameter Weibull distribution, with the location (or origin) parameter unknown, and possibly also the shape and/or scale parameters unknown. Tests of fit are described, and tables are given for the EDF statistics A^2, W^2 and U^2, to make the tests. Several examples are discussed.

Key words. EDF tests: empirical distribution function: goodness-of-fit: reliability: survival analysis.

ESTIMATION AND TESTS OF FIT FOR THE THREE-PARAMETER WEIBULL DISTRIBUTION

1. INTRODUCTION

In this article are given estimation procedures and tests of fit (based on the empirical distribution function, or EDF), for the three-parameter Weibull distribution:

$$F(x;\alpha,\beta,m) = 1 - \exp[-\{(x-\alpha)/\beta\}^m], \qquad x > \alpha \tag{1}$$

where β and m are positive constants. When α is known, the distribution is called the two-parameter Weibull distribution, and estimation procedures and goodness-of-fit tests are then very straightforward: the tests are referenced in Section 2 below. Here we concentrate on tests for use when α is not known, but must be estimated from the sample, together with m and β if necessary. In these circumstances, problems can sometimes arise in estimating the parameters, so in Section 2, procedures are given to obtain efficient estimates which can then be used with the goodness-of-fit tests. For reasons of space many details of both estimation and goodness-of-fit procedures have been omitted; some of these are in Lockhart and Stephens (1989).

It is worthwhile to observe that the three-parameter Weibull distribution is a member of a wider class, the generalized Extreme-Value (or Jenkinson) distribution. This distribution is

$$F^*(x;a,b,c) = 1 - \exp\left[-\left\{1 + c\left(\frac{x-a}{b}\right)\right\}^{1/c}\right], \qquad x > a \tag{2}$$

The parameter b must be positive while a and c may be any real numbers. The three-parameter Weibull distribution is the subfamily of F^* with $c > 0$. The special case $c=0$ is the usual extreme-value distribution

$$F^*(x;a,b,0) = 1 - \exp[-\exp((x-a)/b)], \qquad -\infty < x < \infty; \tag{3}$$

it arises as the limit of the three-parameter Weibull family (1) as $m \to \infty$.

Estimation of parameters. The goodness-of-fit procedures depend on first estimating the parameters in (1) by an efficient method, for example, by maximum likelihood. If one had an infinitely large Weibull sample, such estimates could be found straightforwardly. However, for finite samples, usually small, there will be some data sets which give no local maximum for the likelihood. If one were willing to fit the larger family (2), maximum likelihood estimates could be found, but with a negative c; thus the resulting fitted distribution will not be in the Weibull family. However, practical workers in many fields do not wish to broaden the class of distributions beyond the Weibull, to include (2); then, for practical purposes, the best Weibull fit will be the distribution with $c=0$ (the nearest non-negative c), that is to say, the extreme-value distribution (3). Note that $c=0$ corresponds to $m=\infty$ in (1) (and $\alpha \to -\infty$ and $\beta \to \infty$).

In Section 2 we discuss estimation, and how to recognize this difficult case. A formula is given to calculate, from a given data set, a quantity Δ, which, when negative, indicates that no local maximum for the likelihood can be found, and distribution (3) should be fitted to the data.

Cheng and Iles (1990) have discussed estimation problems for three-parameter distributions embedded in a larger model and in particular, problems for the three-parameter Weibull distribution, and have pointed out the connections between (1) and (2). They offer tests for $c=0$, with the assumption that, if this can be accepted, the extreme-value distribution (3) will be fitted to a data set; if a zero c is rejected, estimation continues for the Weibull distribution. The implication is that, where acceptable, the distribution (3) will be fitted in preference to (1). We are concerned with providing tests of fit for the distribution (1), but estimation of parameters must come first; we concentrate therefore on fitting the Weibull distribution whenever this is possible, and only fall back on (3) where we must assume $\hat{m} = \infty$. There are naturally many points of contact with Cheng and Iles (1990), as we shall see below, although these authors do not discuss tests of fit.

Apart from the problem of deciding whether $\hat{m}$ is infinite, it is well-known that there are other problems of estimation when α is not known; for example, the likelihood can be made infinite if m also is unknown, or if known $m < 1$. An extensive literature exists on this type of problem (for example Smith, 1985; Cheng and Iles 1987). Smith (1985) discusses asymptotic procedures in detail, and gives extensive theory, to which we refer below; Smith and Naylor (1987) compare Bayesian and maximum likelihood estimators in a case study. Of the results which are known and proved, most of them concern asymptotics, where the sample size is infinite, or they concern the estimation situation for m large. In this article we concentrate on the practical cases of finite samples, and smaller values of m.

Goodness-of-fit tests. Suppose a random sample $x_1, x_2, \ldots, x_n$ is given. The goodness-of-fit tests are based on the empirical distribution function (EDF) of the x_i, and in practice the statistics are calculated from the values z_i given by the probability integral transform $z_i = F(x_i\colon \alpha, \beta, m)$ where estimates of α, β and m are used in $F(\cdot)$ when the parameters are not known. EDF tests have been shown to be powerful in many test situations; rival procedures, such as correlation tests or spacings tests, often have zero asymptotic efficiency relative to EDF tests (see, for example, Cibisov (1961) for ARE of spacings tests, and McLaren and Lockhart (1987) for ARE of correlation tests).

The estimation techniques depend on the profile likelihood of the data. They are set out in Section 2. The tests of fit are described in Section 3, and the theory is given in Section 4. Throughout the paper, the plots of the profile likelihood, the estimation procedures and the tests of fit will be illustrated by reference to three data sets, given in Appendix 1. The three sets are as follows:

(a) Data set 1, from Cox and Oakes (1984, Table 1.3) consists of 10 values of the number of cycles to failure when springs are subjected to various stress levels. For these data, the stress level is 950 N/mm^2, and the values are in units of 1000 cycles.

(b) Data set 2 consists of 15 times to failure of air conditioning equipment in aircraft: these are taken from a table of times for several aircraft, given in Proschan (1963, Table 1), and are the data for aircraft number 7910. This set has been used by Stephens (1986b) in studying various tests for exponentiality, and the conclusion was drawn that the times come from a distribution with decreasing failure rate (DFR); if this distribution is Weibull, DFR implies $m < 1$. Clearly both data sets 1 and 2 have been rounded, so they become discrete, but this makes negligible difference to the estimation procedures or the tests of fit.

(c) Data set 3 is artificially constructed, to illustrate the third possible situation which can occur (although more rarely) in analysing a sample.

In order to save space, we give tables only for the EDF statistic A^2, which is known in many situations to have good power. Tables for the statistics W^2 and U^2 are given by Lockhart and Stephens (1989).

Finally we remark that tables for testing fit to the larger family (2) above will be published separately.

2. ESTIMATION PROCEDURES

2.1 The different Cases. For the test of H_o, eight cases can be distinguished, according to which parameters in (1) must be estimated; the other parameters are assumed known. The cases are:

Case 0: all parameters known; Case 1: α unknown; Case 2: β unknown;

Case 3: α and β unknown; Case 4: m unknown; Case 5: α and m unknown; Case 6: β and m unknown; Case 7: α, β and m all unknown.

Case numbers 0,1,2 and 3 correspond to the numbers used for other distributions involving only location and scale parameters; see, for example, Stephens (1986a). In Case 0, the z_i given by the probability integral transform are, on H_0, uniform between 0 and 1; EDF tests for this Case are given in Stephens (1986a, Section 4.4). In Cases 2, 4, and 6 above, where α is known, the transformation $y = -\log(x-\alpha)$ is made and the y-sample is tested to come from the extreme-value distribution

$$F(y) = \exp[-\exp(-(y-\alpha')/\beta')], \qquad -\infty < y < \infty \tag{4}$$

(Throughout the paper, log refers to natural logarithm). EDF tests are given in Stephens (1977, or 1986a, Section 4.11). Here the relationship between (1) and (4) is that $\alpha' = -\log\beta$ and $\beta' = 1/m$.

Thus in this article it is necessary to give tests only for Cases 1, 3, 5 and 7, where α must be estimated, and possibly also β and m. As was stated in Section 1, this is usually done by maximum likelihood, and we first examine the likelihood equations.

Suppose $L(\alpha,\beta,m)$ is the likelihood for a sample $x_1,\dots,x_n$ from (1). The log-likelihood function is then

$$\lambda = \log L(\alpha,\beta,m) = n \log m + (m-1)\, \Sigma\log(x_i-\alpha)$$
$$- nm \log \beta - \Sigma\{(x_i-\alpha)/\beta\}^m \tag{5}$$

where sums are for i from 1 to n. From (5) the likelihood equations are

$$\frac{\partial\lambda}{\partial m} = n/m + \Sigma \log(x_i-\alpha) - \{\Sigma(x_i-\alpha)^m \log(x_i-\alpha)\}/\beta^m -$$
$$(\log \beta)\{n - \Sigma(x_i-\alpha)^m/\beta^m\} = 0 \tag{6}$$

$$\frac{\partial\lambda}{\partial \alpha} = (m-1)\, \Sigma(x_i-\alpha)^{-1} - \{m\, \Sigma(x_i-\alpha)^{m-1}\}/\beta^m = 0 \tag{7}$$

$$\frac{\partial\lambda}{\partial\beta} = -\frac{nm}{\beta} + m\frac{\Sigma(x_i-\alpha)^m}{\beta^{m+1}} = 0 \tag{8}$$

These will be used to give estimates whenever α, m, and/or β are unknown. Equation (8) may be used to eliminate β from (6) or (7), giving

$$\frac{1}{m} - \frac{\Sigma(x_i-\alpha)^m\log(x_i-\alpha)}{\Sigma(x_i-\alpha)^m} + \frac{\Sigma\log(x_i-\alpha)}{n} = 0 \tag{9}$$

$$\frac{m-1}{m}\Sigma(x_i-\alpha)^{-1} + n\frac{\Sigma(x_i-\alpha)^{m-1}}{\Sigma(x_i-\alpha)^m} = 0; \tag{10}$$

also (8) can be written

$$\beta = \left\{\frac{\Sigma(x_i-\alpha)^m}{n}\right\}^{1/m} \tag{11}$$

to give the estimate of β from the estimated $\hat{\alpha}$ and the known or estimated value of m.

The problems of estimation arise because, when m or $\hat{m}$ is less than 1, the likelihood can be made infinite by setting $\hat{\alpha} = x_{(1)}$, where $x_{(1)} < x_{(2)} \cdots < x_{(n)}$ are the order statistics of the sample. This is clearly a biased estimate of α, and we propose a better estimate below.

2.2 Cases 1 and 3: m known.

Suppose the known value of m is m_0; we must discuss the two cases (a) $m_0 > 1$ (b) $0 < m_0 < 1$. (Note that when $m_0 = 1$, the distribution (1) reduces to the exponential distribution with unknown origin; tests for this distribution have been given by Spinelli and Stephens, 1986).

(a) Suppose $m_0 > 1$. Then in Case 1, with β known, equation (7) is solved for $\hat{\alpha}$. In Case 3, (9) is solved for $\hat{\alpha}$ and then (11) for $\hat{\beta}$, using $\hat{\alpha}$. It is easy with computers to solve for these estimates: a straightforward procedure is to start with an estimated $\alpha_t = x_{(1)} - \epsilon$, where ϵ is very small, and to decrease α_t steadily until a solution is found.

(b) Suppose $0 < m_0 < 1$. The M.L. estimate of α is now $\hat{\alpha} = x_{(1)}$; it is a biased estimate and gives an infinite likelihood. We propose the following alternative estimates. Let $k = 1/m_0$. In Case 1, take $\hat{\alpha} = x_{(1)} - \beta/n^k$.

For Case 3, proceed iteratively as follows: Suppose $\beta(\alpha,m)$ is the solution of (11), and start with $\alpha_0 = x_{(1)}$ and $\beta_0 = \beta(\alpha_0,m_0)$. Then take $\alpha_{r+1} = x_{(1)} - \beta_r/n^k$, and $\beta_{r+1} = \beta(\alpha_r,m_0)$, for $r=0,1,\ldots$until α_r, β_r converge to estimates $\hat{\alpha}$ and $\hat{\beta}$. Many studies indicate that this procedure, which we call procedure A, converges rapidly to estimates which usually give a better fit than the usual $\hat{\alpha} = x_{(1)}$, and in Case 3, $\hat{\beta} = \beta(\hat{\alpha};m_0)$. Procedure A is illustrated by Example 2 below.

Example 1. Case 1. Consider data set 1, in Table 1. To illustrate Case 1, suppose values $\beta = 70$ and $m = 2.1$ are assumed known. (These are reasonably consistent with the values estimated in Case 7 below). The M.L.E. of $\hat{\alpha}$ is 105.31, from equation (7).

Example 2, Case 3. Consider data set 2 in Table 1, and suppose $m = 0.5$, say. Then usual estimates are $\hat{\alpha} = x_{(1)} = 12.0$, and $\hat{\beta} = 69.42$, from (11). Procedure A converges rapidly to give $\hat{\alpha} = 11.69$ and $\hat{\beta} = 70.45$.

2.3 Cases 5 and 7.

In these cases, α and m are both unknown. Again the likelihood can be made infinite, by allowing $\hat{\alpha} = x_{(1)}$, and using any estimate $\hat{m}$ such that $\hat{m} < 1$. Thus $\hat{m}$ cannot be made precise. We shall suggest estimates based on the profile likelihood, $L^*(\alpha_t)$, abbreviated L^*. This is the likelihood L maximised with respect to β and m, for a fixed α_t. Thus $L^*(\alpha_t) = L(\alpha_t,\beta_t,m_t)$ where, for a given α_t, for Case 5, β_t is the known β, and then m_t is the solution of (6); for Case 7, m_t and β_t are the solutions of (9) and (11). Case 5 (where only β is known) occurs very rarely in practice, and from here on we shall discuss estimation only for the important Case 7, where all three parameters are unknown.

Case 7. We consider the possible forms which a plot of $Z(\alpha_t) = \log L^*(\alpha_t)$, against α_t, might take. Suppose $Z(\alpha_t)$ is abbreviated to Z. There appear to be only three possible types of plot, illustrated by Figures 1a, 1b and 1c. These are the plots for data sets 1, 2, and 3 respectively.

Figure 1a It may be seen that only Figure 1a has a local maximum, and only with this plot is there a true ML solution, occurring for the value of $\hat{\alpha}$ equal to the α_t at the maximum; then $\hat{m}$ can be found from (9) or (10), and $\hat{\beta}$ from (11) using $\hat{\alpha}$ and $\hat{m}$. The minimum for Z which occurs in Figure 1a gives a saddlepoint for the likelihood. Figure 1a is most likely to arise when the true $m > 1$; if the sample were infinite, it would certainly be the plot arising when $m > 1$. Rockette, Antle and Klimko (1974) have conjectured that Figure 1a occurs with probability converging to 1 as $n \to \infty$ when $m > 1$. They noted that the existence of a local maximum implies the existence of a saddlepoint. Smith (1985) shows *inter alia* that when $m > 1$, there is, with probability approaching 1 as $n \to \infty$, a root of the likelihood equations which is a consistent local maximum of the likelihood.

Figure 1b In Figure 1b, Z steadily decreases as α_t decreases; this figure occurs for finite samples with increasing probability as m becomes smaller; for an infinite sample we believe that it would be certain to occur whenever $m < 1$. The maximum likelihood estimate of α would then be $x_{(1)}$, giving an infinite likelihood for $m < 1$. This estimate is clearly biased, and we recommend the estimate $\hat{\alpha}$ given by Procedure 1b in Section 2.6 below; this gives efficient estimates, with a less biased estimate of α.

Figure 1c Here there is a minimum for Z (again this gives a saddlepoint for the likelihood), but there is no maximum and therefore no ML solution. This figure would effectively never occur if the sample were infinite, (see Smith, 1985) but with finite samples it arises with increasing probability as m increases (see Table 4 below). The graph of Z tends to a horizontal asymptote as $\alpha_t \to -\infty$, suggesting that the MLE of $\alpha_t \to -\infty$; in turn $\hat{m}$ and $\hat{\beta}$ both approach ∞. The Weibull distribution then takes the limiting form discussed in the introduction, namely the extreme-value distribution (3); the relationship between the parameters in (3) and those in (1) is that $b = \lim \beta/m$ and $a = \lim(\alpha+\beta)$, as $\alpha \to -\infty$, $\beta \to \infty$, and $m \to \infty$.

Clearly if it could be decided immediately from a data set that Figure 1c would arise, there would be no need to try to find the MLE for (1), and (3) could be fitted directly. In Section 2.5 we show how this may be done by calculating a quantity Δ whose sign immediately detects Figure 1c, but first we discuss the M.L. equations (9) and (10).

2.4 Solution of the Maximum likelihood equations (Case 7).

Maxima and minima in Figures 1a, 1b and 1c correspond to solutions of (9) and (10) for $\hat{\alpha}$ and $\hat{m}$, and then (11) for $\hat{\beta}$. Let α_t be a trial value for α, and let m_9 be the solution of (9) and m_{10} the solution of (10); these may be found iteratively. If a plot of m_9 against α_t intersects a plot of m_{10} against α_t, we have the solution(s) required. Plots of m_9 and m_{10} against α_t are shown in Figures 2a, 2b and 2c; they are the plots for data sets 1, 2 and 3 respectively. There are three possibilities; no intersection, so no MLE, as in Figure 2b; one intersection, also no MLE, so fit the extreme-value distribution, as in Figure 2c; or two intersections, with MLE corresponding to the solution with smaller α, as in Figure 2a.

2.5 Detection of Figures 1c and 2c We first give a procedure to decide if Figures 1c and 2c are appropriate for a given data set, without actually plotting them. In Figures 2a, 2b, 2c it appears that the solutions for m_9 and m_{10} tend to parallel lines as $\alpha_t \to -\infty$; this is shown to be true in Appendix 2.

Suppose Δ is the limiting "gap" between these lines, that is, $\Delta = \lim_{\alpha_t \to -\infty} (m_{10}-m_9)$; the value of Δ is found as follows. Let $\bar{x} = \Sigma x_i/n$, and $s = \Sigma x_i^2/n$; also define $T_r = \Sigma(x_i)^r \exp(-\gamma x_i)$; in these expressions the sums run for $i=1,\ldots,n$. Let γ be the solution of

$$\frac{1}{\gamma} = \bar{x} - \frac{T_1}{T_0} \qquad (12)$$

The value of γ can easily be found by iteration, starting, for example, with $\gamma = 1$ in the right-hand side terms T_0 and T_1. Define

$$D = \bar{x}T_0 + \gamma(T_2-\bar{x}\,T_1) \;; \qquad (13)$$

γ is the limiting slope of the lines, and Δ is given by

$$\Delta = \{\bar{x}\,T_0 - \gamma(sT_0 - T_2)/2\}/D \qquad (14)$$

It is clear that a negative value of Δ implies Figure 2c, and then the Weibull fit should be abandoned in favour of the extreme-value fit (3) - note that the x-values are fitted directly, and logarithms are not taken first, as in the test with known α in Section 2. The detection of Figures 1c and 2c by Δ is equivalent to use of a discriminant L, proposed by Cheng and Iles (1990) although they do not discuss the behaviour of equations (9) and (10) in the same detail.

Example 3, Case 7. Data set 3 has been constructed to illustrate Figure 1c. When the extreme-value distribution (3) is fitted (see, for example, Stephens, 1977, or 1986a, Section 4.10), the parameter estimates are $\hat{a} = 1.335$, and $\hat{b} = 0.343$. In Section 3, we describe the test of fit for this distribution. It will *not be the same* as that described in Stephens (1977; 1986a, Section 4.10) because it must be made conditionally on the occurrence of Figure 1c.

2.6. Figures 1b and 2b. Suppose Δ is positive, so that either Figure 1a (2a) or 1b(2b) is appropriate, and suppose a plot gives Figure 1b. The conventional estimate will be $\hat{\alpha} = x_{(1)}$, with an infinite likelihood, but again we propose Procedure A, adapted for unknown m, to give a less biased estimate of α, and also, almost always, a better overall fit.

Suppose α_r, β_r, m_r are estimates (we omit the ^ symbol) at iteration r: find estimates α_{r+1}, β_{r+1}, m_{r+1} as follows:

(a) Let $\alpha_{r+1} = x_{(1)} - \beta_r/n^k$, where $k = 1/m_r$.

(b) Then solve (6) for m_{r+1}, using $\alpha = \alpha_{r+1}$;

(c) Use (11) to give β_{r+1}, using α_{r+1} and m_{r+1}.

Iteration of steps (a) to (c) continues until the accuracy required for $\hat{m}$ is obtained. Initial estimates m_0 and β_0 may be found by setting $\alpha_0 = x_{(1)}$ and continuing with steps (b) and (c) above, but using only the n-1 values $x_{(2)}, x_{(3)}, \ldots x_{(n)}$. The final estimates will be the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{m}$ for Figure 1b.

Example 4, Case 7 Figure 1b is the plot for data set 2; the initial estimates α_o, β_o, m_o are then 12.0, 101.01 and $\hat{m} = 0.795$; with Procedure A, 7 iterations give final estimates $\hat{\alpha} = 9.313$, $\hat{\beta} = 93.50$, $\hat{m} = 0.763$. Here iteration was stopped when two successive values of $\hat{m}$ differed by less than 0.001. We shall see in Section 3 that the second set of estimates gives a better fit to distribution (1).

2.7 Figures 1a and 2a: **Example 5, Case 7**: Finally we turn to solutions based on Figures 1a and 2a. These can almost always be found very straightforwardly by solving (9) and (10) by iteration and searching for the crossing in Figure 2a with smaller α_t. For data set 1, the maximum in Figure 1a, and the crossing in Figure 2a, occur at $\hat{\alpha} = 99.02$, with $\hat{m} = 2.38$ and $\hat{\beta} = 78.23$. (The saddlepoint is at $\hat{\alpha}_t = 116.960$, $\hat{m} = 1.008$).

2.8 **Comment**. When Figure 1b occurs, we have found, from many Monte Carlo studies, that Procedure A <u>always</u> converges, and gives a better fit to (1); then $\hat{m}$ is small. For larger values of the true m, Procedure A also converges for some data sets, but there is also an MLE because the plot is like Figure 1a. In this case, the MLE should be taken.

If one is seated before a graphics terminal, it is not difficult to plot the profile likelihood and decide between the various Figures: also to plot m_9 and m_{10} and find the estimates. It is much more difficult to automate the procedure for a computer. The value of Δ in Section 2.5 will decide whether or not Figures 1c and 2c obtain: the problem is to decide between Figures 1a (or 2a) and 1b (or 2b). We suggest that α_t be started so close to $x_{(1)}$ that $m_{10} - m_9$ is almost 1.00 (for the examples given, this may mean $x_{(1)} - \alpha_t$ of the order of 10^{-10}). Then make α_t smaller so that $m_{10} - m_9$ also gets smaller, and either passes through zero (so that a saddlepoint exists and we have figures 1a and 2a), <u>or</u> is clearly seen to approach its limit Δ <u>without</u> passing through zero (and we have Figures 1b and 2b). In the former case, once the saddlepoint has been found, the steps in decreasing α_t can be made larger till the MLE is found when $m_{10} - m_9$ is zero for the second time.

3. <u>GOODNESS-OF-FIT TESTS</u>.

In this section the EDF tests are described. The null hypothesis is

H_0: the random sample $x_1, x_2, \ldots, x_n$ comes from distribution (1).

(a) Find the estimates of unknown parameters as described above, and make the transformation, for $i = 1,2,\ldots,n$, $Z_{(i)} = F(x_{(i)}; \alpha,\beta,m)$, using the estimates where necessary.

(b) Calculate statistics A^2, W^2, U^2 as follows:

$$A^2 = -n - (1/n)\Sigma(2i-1)[\log(z_{(i)}) + \log(1 - z_{(n+1-i)})]$$

$$W^2 = \Sigma(z_{(i)} - (2i-1)/(2n))^2 + 1/(12n).$$

$U^2 = W^2 - n(\bar{z} - 0.5)^2$, where

$\bar{z} = \Sigma z_{(i)}/n$, and where sums are for $i = 1$ to n, and log means natural logarithms.

(We shall illustrate using only A^2 in the Examples below).

(c) Let $c = 1/m$, or $1/\hat{m}$ when m is estimated. Enter Table 1, using the subtable for the appropriate case. When m or $\hat{m} > 2$, we have $0 < c < 0.5$ and the subtable is entered on the line corresponding to c; when m or $\hat{m}$ is less than or equal to 2, so that $c \geq 0.5$, the *last line* of the subtable, labelled $c = 0.5$, should be entered. H_0 is rejected at significance level p if the statistic used is greater than the value given for level p. The table has been given using c rather than m because linear interpolation for c will give good accuracy. In all the tables, the given points are for the asymptotic distributions of the statistics: however, they can be used with good accuracy for smaller values of n (say $n \geq 10$); for $n < 10$ a goodness-of-fit test would in any case have very little power.

Example 1 Case 1 (continued) For data set 1, with m assumed to be 2.1, (so $c = 1/m = 0.476$), $\beta = 70$, and $\hat{\alpha} = 105.31$, the value of A^2 is 0.307. Reference to Table 1 for Case 1, with entry at $c = 0.476$, shows no significance at the 50% level, so the Weibull fit is good.

Example 2. Case 3 (continued) For data set 2, with $m = 0.5$, $\hat{\alpha} = 11.69$ and $\hat{\beta} = 70.44$ as in Example 2, we have $A^2 = .754$. Since $c = 1/m = 2.0$, greater than 0.5, the table for Case 3 is entered on the last line ($c = 0.5$); A^2 is not significant at the 20% level, so the fit is good.

Example 3. Case 7 For data set 3, the extreme-value distribution (3) must be fitted and estimates are $\hat{a} = 1.335$ and $\hat{b} = 0.343$. After the Probability Integral Transform $z_{(i)} = F(x_{(i)})$, where $F(x)$ is given by (3), the value of A^2 is 0.561. This value should then be referred to Table 1, Case 7, with $c = 0$, corresponding to $\hat{m} = \infty$. The p-value is approximately 0.08.

Comment. This test should not be confused with the usual test for the extreme-value distribution with unknown parameters, given by Stephens (1977; 1986a, Section 4.10). The test given in the previous section is made conditionally on the occurrence of Figure 1c for the profile likelihood plot, and Table 3 given here, with $c = 0$, is then the relevant table. The tests given by Stephens are those for the situation where it is intended *from the start* to fit an extreme-value distribution to the data set.

Example 4 Case 7 (continued) For data set 2, assuming all three parameters estimated in the conventional way, giving $\hat{\alpha} = 12.00$, $\hat{\beta} = 101.01$, $\hat{m} = 0.795$, we have $A^2 = 0.703$. When Procedure A is used, the estimates, given in Section 2.10, are now $\hat{\alpha} = 9.313$, $\hat{\beta} = 93.50$, $\hat{m} = 0.763$, and the test statistic is $A^2 = 0.54$, indicating a better fit. Since $\hat{c} = 1/\hat{m} = 1.31$ is greater than 0.5, the Table for Case 7 is entered on the last line ($c = 0.5$). The significance level is then 0.17.

Example 5 *Case 7* (continued) For data set 1, with all three parameters estimated as in Example 3 above, namely $\hat{\alpha} = 99.02$, $\hat{\beta} = 78.23$, and $\hat{m} = 2.38$, $A^2 = 0.260$. Table 1 is entered at $\hat{c} = 1/\hat{m} = 0.420$. The above test value is not significant at the 50% level, so the fit is very good.

4. ASYMPTOTIC THEORY OF EDF TESTS.

4.1 Asymptotic distributions

In this section the asymptotic theory of EDF tests is summarized. The calculation of asymptotic distributions of EDF statistics follows a well-known procedure (see, for example, Durbin, 1973; Stephens, 1976). It is based on the fact that $y_n(z) = \sqrt{n}(F_n(z) - z)$, where $F_n(z)$ is the EDF of the z-set, tends to a Gaussian process y(z) as $n \to \infty$, and the statistics are functionals of this process. The mean of $y(z)$ is zero: we need the covariance function $\rho(s,t) = E\{y(s)\ y(t)\}$. When all parameters are known

(Case 0), this covariance is $\rho_0(s,t) = \min(s,t) - st$. When parameters are estimated, the covariance will depend on those which are estimated; if the method of estimation is efficient the covariance will not depend on true values of location or scale parameters α and β, but will depend on the shape parameter m. We illustrate the calculation for Case 7, the most difficult case.

Suppose the parameters are components of a vector θ: $\theta_1 = \alpha$, $\theta_2 = \beta$, $\theta_3 = m$, and (Case 7) suppose all three are unknown. Let $F(x;\theta)$ now denote the distribution $F(x;\alpha,\beta,m)$ and let $f(x;\theta)$ be the corresponding density. Suppose a vector $g(s)$, with components $g_i(s)$ is constructed as follows:

$$g_i(s) = \frac{\partial F(x;\theta)}{\partial \theta_i}, \qquad i=1,2,3, \tag{15}$$

where the right hand side is written as a function of s using the transformation $s = F(x;\theta)$. Let $\{g(s)\}'$ denote the transpose of $g(s)$. Let D be the (symmetric) matrix with entries $\delta_{ij} = E(-\partial^2 \log f(x;\theta)/\partial\theta_i\partial\theta_j)$, $i,j = 1,2,3$, where E denotes expectation, and let Σ be the inverse of D. Then, for Case 7,

$$\rho_7(s,t) = \rho_0(s,t) - \{g(s)\}'\Sigma\, g(s) . \tag{16}$$

From (23) above, the components of $g(s)$ become, after some algebra, and using F for $F(x;\theta)$:

$$\left.\begin{aligned} g_1(s) &= \frac{\partial F}{\partial x} = -\frac{m(1-s)}{\beta}\cdot\{-\log(1-s)\}^{(m-1)/m} \\ g_2(s) &= \frac{\partial F}{\partial \beta} = \frac{m(1-s)}{\beta}\log(1-s) \\ g_3(s) &= \frac{\partial F}{\partial m} = -\frac{(1-s)}{m}\{\log(1-s)\}\log\{-\log(1-s)\} \end{aligned}\right\} \tag{17}$$

Also, for Case 7, and for $m > 2$, D has the top right terms:

$$D = \begin{bmatrix} \frac{(m-1)^2}{\beta^2}\Gamma\left(1-\frac{2}{m}\right) & \vdots & \frac{m(m-1)}{\beta^2}\Gamma\left(1-\frac{1}{m}\right) & \vdots & \frac{\Gamma(m-1)-\Gamma\left(2-\frac{1}{m}\right)-\Gamma'\left(2-\frac{1}{m}\right)}{\beta} \\ & \vdots & \frac{m^2}{\beta^2} & \vdots & -\frac{(\Gamma'(1)+1)}{\beta} \\ & & & \vdots & \frac{\Gamma''(1)+2\Gamma'(1)}{m^2} \end{bmatrix} \tag{18}$$

When Σ is calculated and $g(s)$ and Σ are inserted into (16) $\rho_7(s,t)$ will be independent of α and β. When $m \le 2$, the M.L. estimate $\hat{\alpha}$ of α is super efficient in the sense of Darling (1955) and then the covariance will not need the first term in $g(s)$ and corresponding terms in D; the asymptotics are the same as if α were known, that is, for Case 6. Thus $g(s) = (g_2(s), g_3(s))$ and D is as in (18) but with the top row and first column removed.

For other cases, $\rho_k(s,t)$ for Case k is calculated using only those components in $g_i(s)$ which correspond to unknown θ_i, with the corresponding entries in D, before this is inverted to give the Σ used in (16). The Cramér-von Mises statistic W^2 is based directly on the process $y(z)$, while A^2 is based on the process $a(z) = y(z)/\{z(1-z)\}^{1/2}$; asymptotically W^2 and A^2 are given by

$$W^2 = \int_0^1 y^2(z)dz \quad \text{and} \quad A^2 = \int_0^1 a^2(z)dz .$$

The asymptotic distributions of both statistics are a sum of weighted independent χ_1^2 variables; the weights must be found from the eigenvalues of an integral equation with, for W^2, the $\rho_k(s,t)$ for case k, as kernel. For A^2 one must find the $\rho_k(s,t)$ of the $a(z)$ process. Once the weights are known, the percentage points of the distributions can be calculated by Imhof's method. The techniques are straightforward once the $\rho_k(s,t)$ are known, and we omit the details: they are given in Lockhart and

Stephens (1989).

5. FURTHER REMARKS

5.1 Asymptotic distributions and Monte Carlo studies The results of Smith (1985) can be used to establish rigorously that W^2 has the asymptotic distributions calculated as above, for any estimator found by a method of estimation having the properties of Theorem 3 of Smith. In an unpublished note the present authors have demonstrated the existence of such an estimator by establishing that it is possible to select the "correct" local maximum of the likelihood in the event that there are several. However, we emphasize that no Monte Carlo data set has ever been found in which two such local maxima exist. We should also note here that, for A^2, we are unable to give a rigorous derivation for the asymptotic distribution; this problem arises whenever parameters are estimated in goodness-of-fit tests of the Anderson-Darling type. The asymptotic distribution is, however, confirmed by extensive Monte Carlo studies. These Monte Carlo studies were undertaken to confirm various features of the above estimation and testing procedures. Typically, they involved 10000 samples of sizes n = 10, 20, 30, 40, 60, 100, and 200, and with values of m from 0.1 to 10. The studies firstly confirmed the conjecture, proposed by other authors also, that plots of m_9 and m_{10} will not cross more than twice, so that a local maximum of the likelihood occurs, if at all, only once. This is clearly important in knowing what to search for when the parameters are to be estimated. The studies also confirmed the success of Procedure A in providing estimates to give a better fit, noted earlier.

5.2. Frequency of Figures 1a 1b and 1c. These studies also revealed how the relative frequency of Figures 1a, 1b and 1c will depend on n and on m. As n grows larger, Figure 1c becomes less and less likely and as m becomes smaller, for fixed n, the relative frequency changes from Figure 1a towards Figure 1b. These results are illustrated in Table 4.

The results of Smith (1985) guarantee that Figure 1c arises only with probability tending to 0 as n tends to infinity for any fixed finite value of m in (1). If the data are sampled from the two parameter extreme value distribution, however, (that is, (1) with $m \to \infty$) Figure 1c may be expected to arise about half the time. This can be explained as follows. The sampling can be regarded as from the family (2), with $c = 0$, and *samples* will give sometimes $\hat{c} > 0$ (Figure 1a), and sometimes $\hat{c} < 0$ (Figure 1c). This problem of unobtainable estimates within the desired family also arises in testing for the von Mises distribution on the circle with known direction of concentration. In Lockhart and Stephens (1985) it was suggested that there too an expansion of the model could overcome this problem. Similarly here, testing for the Generalized Extreme Value distribution (2) rather than the 3 parameter Weibull distribution (1) will eliminate the awkwardness. Tests for the enlarged model (2) will be presented in a future paper.

5.3 Convergence of distributions of EDF statistics For finite n, the Monte Carlo studies show that the distributions of W^2, U^2 and A^2 converge rapidly to the asymptotic. This is similar to the behaviour of these statistics in other test situations and tables will not be given. The asymptotic points in Table 1 can be used with good accuracy for $n \geq 10$.

EDF statistics are known to provide powerful tests for many distributions; the powers naturally depend on the alternatives considered, and a study is being made on power properties for the various alternatives to the Weibull usually encountered. On the whole, with the limited power results at present available, the statistic A^2 above is suggested as the preferred statistic for overall Weibull testing.

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APPENDIX 1: DATA SETS

Set 1. Number of cycles to failure of springs (in units of 1000 cycles)

225	171	198	189	189	135	162	135	117	162

Set 2. Times to failure of airconditioning equipment for an aircraft.

74	57	48	29	502	12	70	21	29	386	59	27	153	26	326

Set 3. Artificial data.

0.273	0.468	0.504	0.535	0.617	0.804	0.932	1.034	1.289	1.293
1.294	1.376	1.399	1.407	1.422	1.497	1.521	1.542	1.685	1.737

APPENDIX 2

(a) Asymptotes for m_9 and m_{10} (Section 2.5)

Suppose, as $\alpha \to -\infty$, the relation $m = \gamma\alpha + b_0 + b_1/\alpha$, where γ, b_0 and b_1 and b_1 are to be determined below for each of m_9 and m_{10}. Then we have

$$\left(1 - \frac{x_i}{\alpha}\right)^m = \exp\left\{-\left(\gamma\alpha + b_0 + \frac{b_1}{\alpha}\right)\left(\frac{x_i}{\alpha} + \frac{x_i^2}{2\alpha^2} + \frac{x_i^3}{3\alpha^3}\right)\right\} \tag{A1}$$

$$= e^{-\gamma x_i}\exp\left\{-\left\{\frac{[\gamma x_i^2/2 + b_0x_i]}{\alpha} + \frac{[\gamma x_i^3/3 + b_0x_i^2/2 + b_1x_i]}{\alpha^2}\right\}\right\}$$

$$= e^{-\gamma x_i}\exp(-Z), \text{ say}, \approx e^{-\gamma x_i}\,[1 - (Z) + (Z^2/2)]$$

$$= e^{-\gamma x_i}\left[1 - \frac{\gamma x_i^2/2 + b_0x_i}{\alpha} + \frac{-\left(\frac{\gamma x_i^3}{3} + \frac{b_0x_i^2}{2} + b_1x_i\right) + \frac{\gamma^2x_i^4}{8} + \frac{b_0^2x_i^2}{2}}{\alpha^2} + \frac{b_0\gamma x_i^3}{2}\right].$$

Hence setting $T_r = \sum x_i^r\, e^{-\gamma x_i}$, where sums are for i from 1 to n, we get

$$\sum_{i=1}^{n}(1 - x_i/\alpha)^m = T_0 + \frac{A_2}{\alpha} + \frac{B_2}{\alpha^2} \quad \text{where}$$

$A_2 = -[\gamma T_2/2 + b_0T_1]$ and

$B_2 = -(\gamma T_3/3 + b_0T_2/2 + b_1T_1) + \gamma^2T_4/8 + b_0\gamma T_3/2 + b_0^2\, T_2/2$.

Also $C_2 = \Sigma(1-x_i/\alpha)^{m-1} = T_0 + \dfrac{A_3}{\alpha} + B_3/\alpha^2$, where

$A_3 = -[\gamma T_2/2 + (b_0-1)T_1]$ and

$B_3 = -(\gamma\, T_3/3+(b_0-1)\, T_2/2 + b_1T_1) + \gamma^2T_4/8 + (b_0-1)\gamma T_3/2 + (b_0^2-2b_0+1)T_2/2$.

We now use the above results in equation (10). This may be written

$$\frac{m-1}{n}\Sigma(1-x_i/\alpha)^{-1} = m\,\Sigma(1-x_i/\alpha)^{m-1} .$$

Carrying three terms in each expression, we have

$$\left(\gamma_\alpha + b_0 - 1 + \frac{b_1}{\alpha}\right)\left(1+\frac{\bar{x}}{\alpha}+\frac{s}{\alpha^2}\right)\left(T_0+\frac{A_2}{\alpha}+\frac{B_2}{\alpha^2}\right) = \left(\gamma\alpha + b_0 + \frac{b_1}{\alpha}\right)\left(T_0+\frac{A_3}{\alpha}+\frac{B_3}{\alpha^3}\right)$$

where $\bar{x} = \Sigma x_i/n$ and $s = \Sigma x_i^2/n$. The coefficients of the constant term give

$\gamma\bar{x}T_0 + (b_0-1)T_0 + \gamma A_2 = \gamma A_3 + b_0T_0$; then $\gamma(\bar{x}T_0 + A_2 - A_3) = T_0$. But

$A_2 - A_3 = -T_1$, so we have for γ,

$$\frac{1}{\gamma} = (\bar{x} - T_1/T_0) . \qquad \text{(A2)}$$

Next, equating coefficients of $1/\alpha$, we have

$$\gamma s\, T_0 + \gamma\,\bar{x}\,A_2 + \gamma B_2 + (b_0-1)\bar{x}T_0 + (b_0-1)A_2 + b_1T_0 = \gamma\,B_3 + b_0A_3 + b_1T_0;$$

then $\quad \gamma(s\,T_0 + \bar{x}A_2 + B_2 - B_3) + b_0(\bar{x}T_0 + A_2 - A_3) - \bar{x}T_0 - A_2 = 0 . \qquad$ (A3)

Here

$$B_2 = -\left(\frac{\gamma T_3}{3}+\frac{b_0T_2}{2}+b_1T_1\right)+\frac{\gamma^2T_4}{8}+\frac{b_0\gamma T_3}{2}+\frac{b_0^2T_2}{2} \quad \text{and}$$

$$B_3 = -\left(\frac{\gamma T_3}{3}+\frac{(b_0-1)T_2}{2}+b_1T_1\right)+\frac{\gamma^2T_4}{8}+\frac{(b_0-1)\gamma T_3}{2}+\frac{(b_0^2-2b_0+1)T_2}{2} ;$$

thus

$B_2 - B_3 = \frac{\gamma T_3}{2} + b_0T_2 - T_2$. Recall also that $A_2 - A_3 = -T_1$; then,

substitution in (A3) gives

$$b_0(\bar{x}T_0 + \gamma\,T_2 - \gamma\bar{x}T_1) + \frac{\gamma^2}{2}(T_3 - \bar{x}T_2) + \gamma(sT_0 - T_2/2)$$
$$- \bar{x}T_0 = 0 \qquad \text{(A4)}$$

Solution of (A2) for γ, and (A4) for b_0, now called δ_{10}, gives the coefficients in the asymptote $m_{10} = \gamma\alpha + \delta_{10}$, given in Section 2.5.

We now consider equation (9), which may be written

$$\Sigma(1 - x_i/\alpha)^m = m\left\{\Sigma \log(1-x_i/\alpha)(1 - x_i/\alpha)^m - \left(\frac{1}{n}\Sigma \log(1-x_i/\alpha)\right)\Sigma(1-x_i/\alpha)^m\right\} .$$

Expanding, we have

$$\left(T_0 + \frac{A_2}{\alpha}\right) = (\gamma\alpha + b_0)\left\{- \Sigma\left(\frac{x_i}{\alpha} + \frac{x_i^2}{2\alpha^2}\right) e^{-\gamma x_i}\left(1 - \frac{(\gamma x_i^2/2 + b_0x_i)}{\alpha}\right) + \frac{B_2}{\alpha^2}\right.$$

$$\left. + \left(\frac{\bar{x}}{\alpha} + \frac{s}{2\alpha^2}\right)\left(T_0 + \frac{A_2}{\alpha}\right)\right\} \text{ ; then}$$

$$\left(T_0 + \frac{A_2}{\alpha}\right) = (\gamma\alpha + b_0)\left\{- \frac{T_1}{\alpha} - \frac{T_2/2}{\alpha^2} + \frac{\gamma\, T_3/2 + b_0T_2}{\alpha^2} + \frac{\bar{x}T_0}{\alpha} + \frac{s\, T_0/2 + A_2\bar{x}}{\alpha^2}\right\}$$

$$= \gamma(\bar{x}T_0-T_1) + \left\{b_0(\bar{x}T_0-T_1) + \gamma[sT_0/2 + A_2\bar{x} + (\gamma T_3-T_2)/2 + b_0T_2]\right\}/\alpha .$$

Equating the constant terms gives $T_0 = \gamma(\bar{x}T_0-T_1)$, the same as for equation 10. Thus the asymptotes for m_9 and m_{10} will be parallel. Further, the coefficient of $1/\alpha$ gives

$$A_2 = b_0(\bar{x}T_0 - T_1) + \gamma\frac{\gamma sT_0}{2} + \gamma A_2\bar{x} - \frac{\gamma T_2}{2} + \frac{\gamma^2T_3}{2} + \gamma b_0T_2 \text{ ;}$$

more algebra gives

$$b_0(\bar{x}T_0 + \gamma T_2 - \gamma T_1\bar{x}) + \gamma^2\left(\frac{T_3}{2} - \frac{T_2\bar{x}}{2}\right) + \frac{\gamma sT_0}{2} = 0 \qquad \text{(A5)}$$

Solution of (A5) for b_0 gives the constant δ_9 in the asymptote $m_9 = \gamma\alpha + \delta_9$ quoted in Section 2.5.

<u>Difference Δ</u> Finally, we have $\Delta = \delta_{10} - \delta_9$ given by

$$(\delta_{10}-\delta_9)(\bar{x}T_0 + \gamma T_2- \gamma\bar{x}T_1) = \bar{x}T_0 - \gamma(sT_0-T_2)/2 ,$$

as in equation (14) of Section 2.5.

APPENDIX 3

In Section 3.1 it was pointed out that the means of the asymptotic distributions of W^2 and U^2 can be found analytically. For completeness, we list below six integrals which arise in these calculations.

$$I_1 = \int_0^1 (1-s)^2 \{-\log(1-s)\}^{2(m-1)/m} ds = \Gamma(3-2/m)/3^{(3-2/m)} \ ;$$

$$I_2 = \int_0^1 (1-s)^2\{\log(1-s)\}^2 ds = 2/27$$

$$I_3 = \int_0^1 (1-s)^2\{\log(1-s)\}^2[\log\{-\log(1-s)\}]^2 \ ds$$

$$= 2\{\pi^2/6 + \gamma^2 - 3\gamma + 2 + (2\gamma-3)\log 3 + \log^2 3\}/27$$

$$= 0.105618$$

$$I_4 = \int_0^1 (1-s)^2\{-\log(1-s)\}^{2-1/m} ds = \Gamma(b)/3^b$$

$$I_5 = \int_0^1 (1-s)^2\{-\log(1-s)\}^{2-1/m} \log\{-\log(1-s)\}ds$$

$$= \{\Gamma'(b) - \log 3 \ \Gamma(b)\}/3^b \ .$$

where, in I_4 and I_5, $b = 3-1/m$;

$$I_6 = \int_0^1 (1-s)^2\{\log(1-s)\}^2 \ \log\{-\log(1-s)\}ds$$

$$= \{\Gamma'(3) - 2 \log 3\}/27$$

$$= \{3 - 2\gamma - 2 \log 3\}/27 = - 0.01302$$

TABLE 1

CRITICAL POINTS FOR WEIBULL CASE 1 FOR A**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.496	0.735	0.915	1.061	1.321	1.590	1.958	2.243
0.05	0.484	0.713	0.882	1.019	1.260	1.510	1.851	2.115
0.10	0.474	0.693	0.853	0.982	1.208	1.440	1.756	2.001
0.15	0.467	0.677	0.830	0.953	1.166	1.385	1.680	1.909
0.20	0.464	0.670	0.818	0.936	1.141	1.350	1.631	1.847
0.25	0.468	0.674	0.821	0.938	1.139	1.343	1.615	1.825
0.30	0.485	0.696	0.847	0.966	1.171	1.377	1.651	1.860
0.35	0.517	0.747	0.910	1.039	1.259	1.480	1.774	1.997
0.40	0.572	0.840	1.032	1.185	1.448	1.713	2.068	2.340
0.45	0.654	0.995	1.249	1.455	1.816	2.189	2.694	3.086
0.50	0.774	1.248	1.621	1.933	2.492	3.077	3.878	4.383

CRITICAL POINTS FOR WEIBULL CASE 3 FOR A**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.342	0.472	0.563	0.635	0.757	0.879	1.043	1.167
0.05	0.345	0.476	0.568	0.640	0.763	0.886	1.051	1.176
0.10	0.349	0.482	0.575	0.648	0.773	0.898	1.065	1.193
0.15	0.354	0.491	0.586	0.661	0.789	0.917	1.088	1.219
0.20	0.368	0.503	0.602	0.679	0.812	0.945	1.122	1.258
0.25	0.374	0.520	0.624	0.705	0.844	0.984	1.171	1.314
0.30	0.388	0.544	0.654	0.740	0.889	1.039	1.240	1.394
0.35	0.407	0.574	0.694	0.788	0.951	1.116	1.338	1.509
0.40	0.430	0.614	0.747	0.853	1.037	1.224	1.478	1.673
0.45	0.459	0.667	0.819	0.941	1.156	1.376	1.675	1.909
0.50	0.496	0.735	0.915	1.061	1.321	1.590	1.958	2.243

CRITICAL POINTS FOR WEIBULL CASE 5 FOR A**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.342	0.472	0.563	0.635	0.757	0.879	1.043	1.167
0.05	0.343	0.473	0.564	0.636	0.758	0.881	1.045	1.170
0.10	0.344	0.476	0.567	0.640	0.763	0.887	1.052	1.178
0.15	0.348	0.480	0.574	0.647	0.772	0.898	1.067	1.195
0.20	0.353	0.489	0.585	0.660	0.789	0.919	1.092	1.225
0.25	0.363	0.505	0.604	0.688	0.818	0.955	1.138	1.278
0.30	0.381	0.532	0.640	0.725	0.872	1.021	1.222	1.377
0.35	0.411	0.583	0.706	0.805	0.977	1.154	1.394	1.580
0.40	0.460	0.673	0.832	0.962	1.192	1.432	1.762	2.017
0.45	0.535	0.828	1.061	1.257	1.611	1.982	2.490	2.883
0.50	0.634	1.059	1.418	1.724	2.276	2.853	3.639	4.247

CRITICAL POINTS FOR WEIBULL CASE 7 FOR A**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.292	0.395	0.467	0.522	0.617	0.711	0.836	0.931
0.05	0.295	0.399	0.471	0.527	0.623	0.719	0.845	0.941
0.10	0.298	0.403	0.476	0.534	0.631	0.728	0.856	0.954
0.15	0.301	0.408	0.483	0.541	0.640	0.738	0.869	0.969
0.20	0.305	0.414	0.490	0.549	0.650	0.751	0.885	0.986
0.25	0.309	0.421	0.498	0.559	0.662	0.765	0.902	1.007
0.30	0.314	0.429	0.508	0.570	0.676	0.782	0.923	1.030
0.35	0.320	0.438	0.519	0.583	0.692	0.802	0.947	1.057
0.40	0.327	0.448	0.532	0.598	0.711	0.824	0.974	1.089
0.45	0.334	0.469	0.547	0.615	0.732	0.850	1.006	1.125
0.50	0.342	0.472	0.563	0.636	0.757	0.879	1.043	1.167

TABLE 2

CRITICAL POINTS FOR WEIBULL CASE 1 FOR W**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.074	0.116	0.148	0.174	0.222	0.271	0.338	0.390
0.05	0.071	0.111	0.140	0.164	0.207	0.251	0.312	0.358
0.10	0.069	0.106	0.134	0.156	0.195	0.235	0.289	0.332
0.15	0.068	0.103	0.129	0.149	0.186	0.222	0.272	0.310
0.20	0.067	0.101	0.126	0.146	0.180	0.215	0.261	0.296
0.25	0.068	0.102	0.127	0.147	0.180	0.214	0.259	0.293
0.30	0.071	0.107	0.133	0.154	0.189	0.224	0.271	0.306
0.35	0.077	0.117	0.147	0.171	0.211	0.252	0.306	0.347
0.40	0.086	0.136	0.172	0.202	0.254	0.308	0.381	0.436
0.45	0.100	0.164	0.215	0.257	0.332	0.410	0.518	0.601
0.50	0.119	0.209	0.284	0.347	0.461	0.581	0.744	0.869

CRITICAL POINTS FOR WEIBULL CASE 3 FOR W**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.050	0.073	0.089	0.102	0.124	0.146	0.175	0.197
0.05	0.051	0.074	0.091	0.104	0.126	0.149	0.179	0.202
0.10	0.052	0.076	0.093	0.107	0.130	0.153	0.184	0.208
0.15	0.054	0.078	0.096	0.110	0.134	0.159	0.191	0.216
0.20	0.055	0.081	0.100	0.115	0.140	0.166	0.201	0.227
0.25	0.057	0.085	0.105	0.121	0.148	0.176	0.213	0.241
0.30	0.060	0.089	0.110	0.128	0.157	0.187	0.228	0.260
0.35	0.063	0.094	0.117	0.136	0.169	0.202	0.248	0.283
0.40	0.066	0.100	0.126	0.147	0.183	0.221	0.272	0.311
0.45	0.070	0.107	0.136	0.159	0.200	0.243	0.301	0.346
0.50	0.074	0.116	0.148	0.174	0.222	0.271	0.338	0.390

CRITICAL POINTS FOR WEIBULL CASE 5 FOR W**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.050	0.073	0.089	0.102	0.124	0.146	0.175	0.197
0.05	0.050	0.073	0.089	0.102	0.124	0.146	0.176	0.198
0.10	0.051	0.074	0.090	0.103	0.126	0.148	0.178	0.200
0.15	0.052	0.075	0.092	0.105	0.128	0.151	0.182	0.206
0.20	0.053	0.078	0.095	0.109	0.133	0.157	0.189	0.214
0.25	0.055	0.081	0.100	0.115	0.141	0.167	0.202	0.229
0.30	0.059	0.088	0.109	0.126	0.155	0.185	0.226	0.257
0.35	0.065	0.098	0.124	0.144	0.180	0.218	0.270	0.310
0.40	0.073	0.116	0.150	0.178	0.228	0.281	0.353	0.409
0.45	0.086	0.145	0.194	0.236	0.311	0.390	0.498	0.581
0.50	0.102	0.186	0.258	0.320	0.431	0.547	0.706	0.827

CRITICAL POINTS FOR WEIBULL CASE 7 FOR W**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.044	0.062	0.075	0.085	0.103	0.120	0.144	0.162
0.05	0.044	0.063	0.076	0.086	0.104	0.122	0.145	0.163
0.10	0.044	0.063	0.077	0.087	0.105	0.123	0.147	0.165
0.15	0.045	0.064	0.077	0.088	0.106	0.125	0.149	0.168
0.20	0.045	0.065	0.079	0.089	0.108	0.127	0.152	0.170
0.25	0.046	0.066	0.080	0.091	0.110	0.129	0.154	0.174
0.30	0.047	0.067	0.081	0.093	0.112	0.132	0.157	0.177
0.35	0.047	0.068	0.083	0.094	0.114	0.134	0.161	0.181
0.40	0.048	0.069	0.085	0.097	0.117	0.138	0.165	0.186
0.45	0.049	0.071	0.087	0.099	0.120	0.141	0.170	0.191
0.50	0.050	0.073	0.089	0.102	0.124	0.146	0.175	0.197

TABLE 3

CRITICAL POINTS FOR WEIBULL CASE 1 FOR U**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.060	0.090	0.111	0.129	0.159	0.189	0.230	0.261
0.05	0.060	0.090	0.111	0.129	0.159	0.189	0.230	0.261
0.10	0.060	0.090	0.112	0.129	0.159	0.189	0.230	0.262
0.15	0.060	0.090	0.112	0.130	0.160	0.190	0.231	0.262
0.20	0.061	0.091	0.113	0.131	0.161	0.191	0.232	0.264
0.25	0.062	0.092	0.114	0.132	0.163	0.193	0.235	0.266
0.30	0.063	0.094	0.116	0.134	0.165	0.196	0.238	0.270
0.35	0.064	0.096	0.119	0.138	0.169	0.201	0.243	0.276
0.40	0.066	0.100	0.124	0.143	0.176	0.209	0.253	0.286
0.45	0.069	0.105	0.131	0.152	0.187	0.222	0.268	0.304
0.50	0.069	0.105	0.131	0.152	0.187	0.222	0.268	0.304

CRITICAL POINTS FOR WEIBULL CASE 3 FOR U**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.048	0.070	0.085	0.097	0.118	0.139	0.167	0.187
0.05	0.049	0.071	0.086	0.098	0.119	0.140	0.168	0.189
0.10	0.049	0.071	0.087	0.100	0.121	0.142	0.170	0.192
0.15	0.050	0.073	0.089	0.101	0.123	0.145	0.174	0.195
0.20	0.051	0.074	0.091	0.104	0.126	0.148	0.178	0.200
0.25	0.052	0.076	0.093	0.107	0.130	0.153	0.184	0.207
0.30	0.054	0.079	0.096	0.111	0.135	0.159	0.191	0.215
0.35	0.055	0.082	0.100	0.115	0.141	0.167	0.201	0.227
0.40	0.058	0.085	0.105	0.121	0.149	0.176	0.214	0.242
0.45	0.060	0.090	0.111	0.129	0.159	0.189	0.230	0.261
0.50	0.060	0.090	0.111	0.129	0.159	0.189	0.230	0.261

CRITICAL POINTS FOR WEIBULL CASE 5 FOR U**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.048	0.070	0.085	0.097	0.118	0.138	0.166	0.187
0.05	0.048	0.070	0.085	0.097	0.118	0.139	0.167	0.188
0.10	0.049	0.070	0.086	0.098	0.119	0.140	0.168	0.189
0.15	0.049	0.071	0.087	0.099	0.121	0.142	0.170	0.192
0.20	0.050	0.072	0.088	0.101	0.123	0.145	0.174	0.196
0.25	0.051	0.074	0.090	0.103	0.126	0.148	0.178	0.201
0.30	0.052	0.076	0.093	0.106	0.130	0.153	0.185	0.209
0.35	0.053	0.078	0.096	0.110	0.135	0.160	0.194	0.220
0.40	0.055	0.082	0.101	0.116	0.143	0.170	0.207	0.235
0.45	0.057	0.085	0.106	0.123	0.152	0.181	0.222	0.253
0.50	0.057	0.085	0.106	0.123	0.152	0.181	0.222	0.253

CRITICAL POINTS FOR WEIBULL CASE 7 FOR U**2

C	0.500	0.750	0.850	0.900	0.950	0.975	0.990	0.995
0.0	0.043	0.061	0.074	0.084	0.102	0.119	0.143	0.160
0.05	0.043	0.062	0.075	0.085	0.103	0.121	0.144	0.162
0.10	0.044	0.062	0.076	0.086	0.104	0.122	0.146	0.164
0.15	0.044	0.063	0.077	0.087	0.105	0.123	0.148	0.166
0.20	0.045	0.064	0.077	0.088	0.107	0.125	0.150	0.168
0.25	0.045	0.065	0.078	0.089	0.108	0.127	0.152	0.171
0.30	0.046	0.065	0.080	0.091	0.110	0.129	0.154	0.173
0.35	0.046	0.066	0.081	0.092	0.111	0.131	0.157	0.176
0.40	0.047	0.067	0.082	0.094	0.113	0.133	0.159	0.180
0.45	0.048	0.068	0.083	0.095	0.115	0.136	0.162	0.183
0.50	0.048	0.070	0.085	0.097	0.118	0.138	0.166	0.187

TABLE 4

RELATIVE FREQUENCY OF FIGURES 1a, 1b and 1c.

The table gives the percentage of 10000 Monte Carlo samples giving the different Figures, for different sample sizes n and different values of c = 1/m.

c	n=20	Figure 1a	Figure 1b	Figure 1c
.1		72.2	0.5	27.3
.25		91.0	1.5	7.5
.5		89.5	9.9	0.6
.67		69.6	30.3	0.1
.83		38.6	61.4	0
1.00		15.0	85.0	0
1.25		2.8	97.1	0
2.00		0	100	
c	n = 40			
.1		81.5	0.0	18.5
.25		98.6	0.0	1.4
.5		99.5	0.5	0
.67		94.4	5.6	0
.83		61.7	38.3	0
1.00		18.7	81.3	0
1.25		1.1	98.9	0
2.00		0	100.0	0
c	n = 60			
.1		87.0	0	13.0
.25		100.0	0	0
.5		99.0	1.0	0
.67		90.8	9.2	0
.83		70.4	29.6	0
1.00		18.4	81.6	0
1.25		0	100.0	0
c	n = 100			
.1		93.1	0	6.9
.25		100.0	0	0
.50		100.0	0	0
.67		100.0	0	0
.83		91.6	8.4	0
1.00		23.9	76.1	0
1.25		0.1	99.9	0
2.00				

Figure 1a
Plot of log profile likelihood for data set 1 (Cox and Oakes)

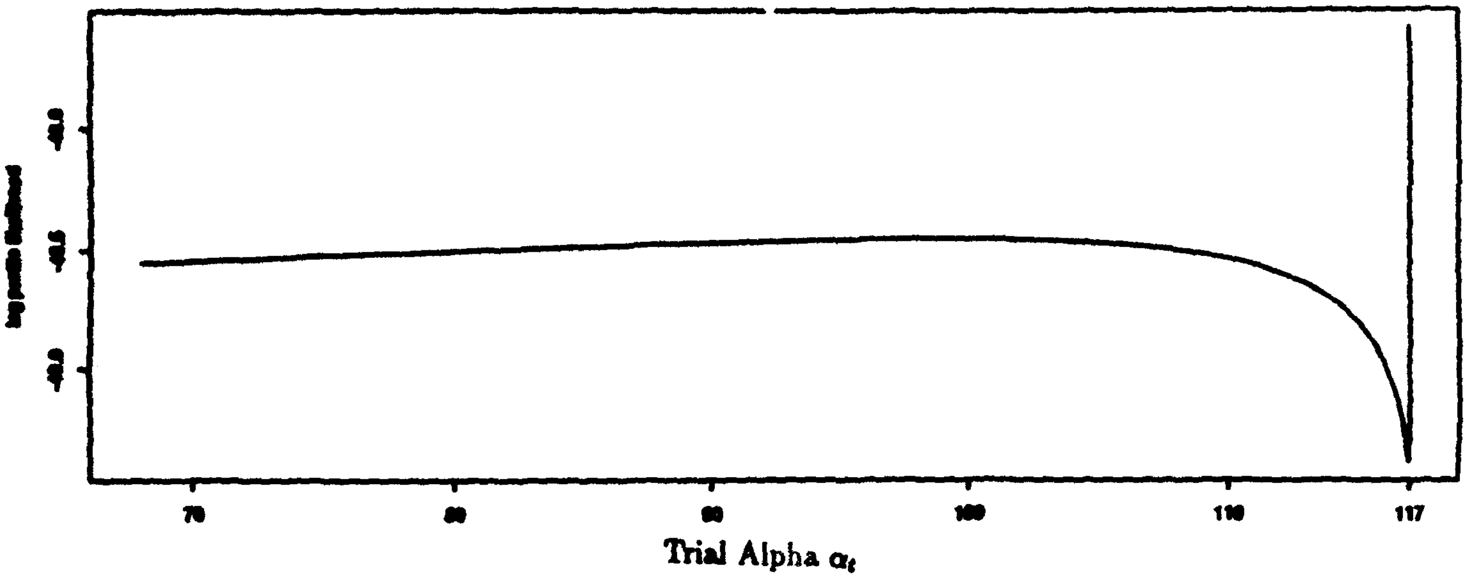


Figure 1b
Plot of log profile likelihood for data set 2 (Proschan)

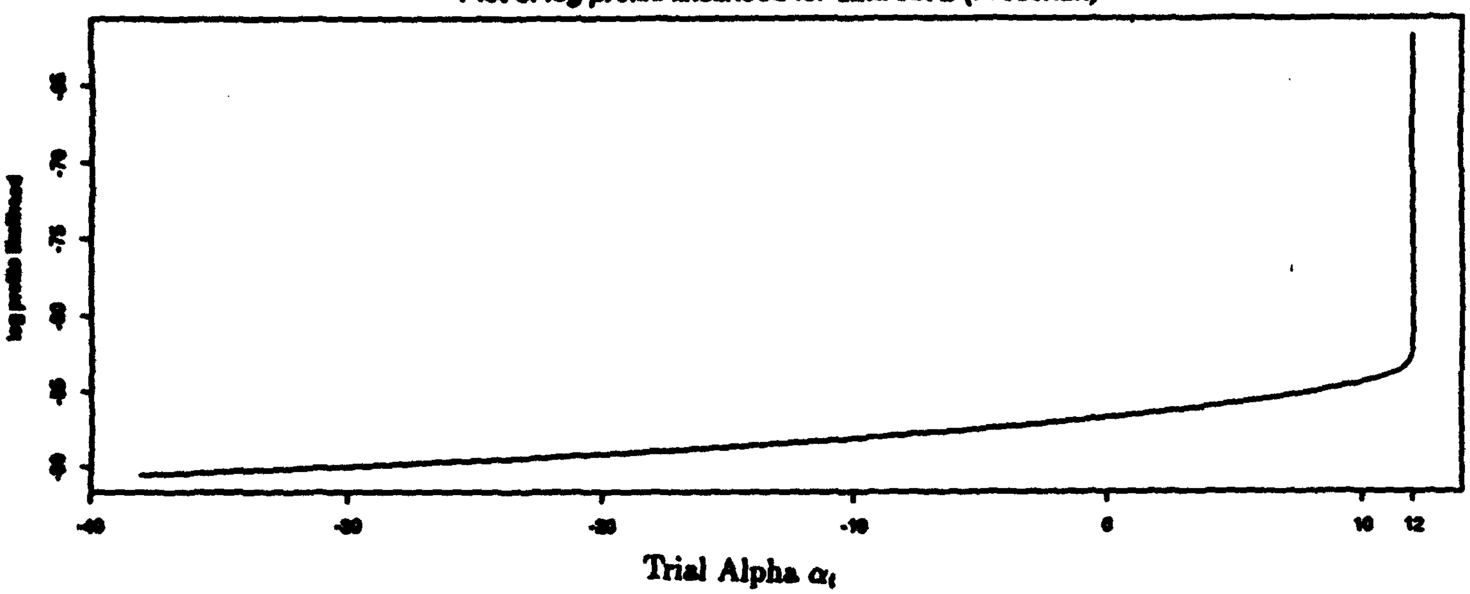


Figure 1c
Plot of log profile likelihood for generated data set 3 (=20)

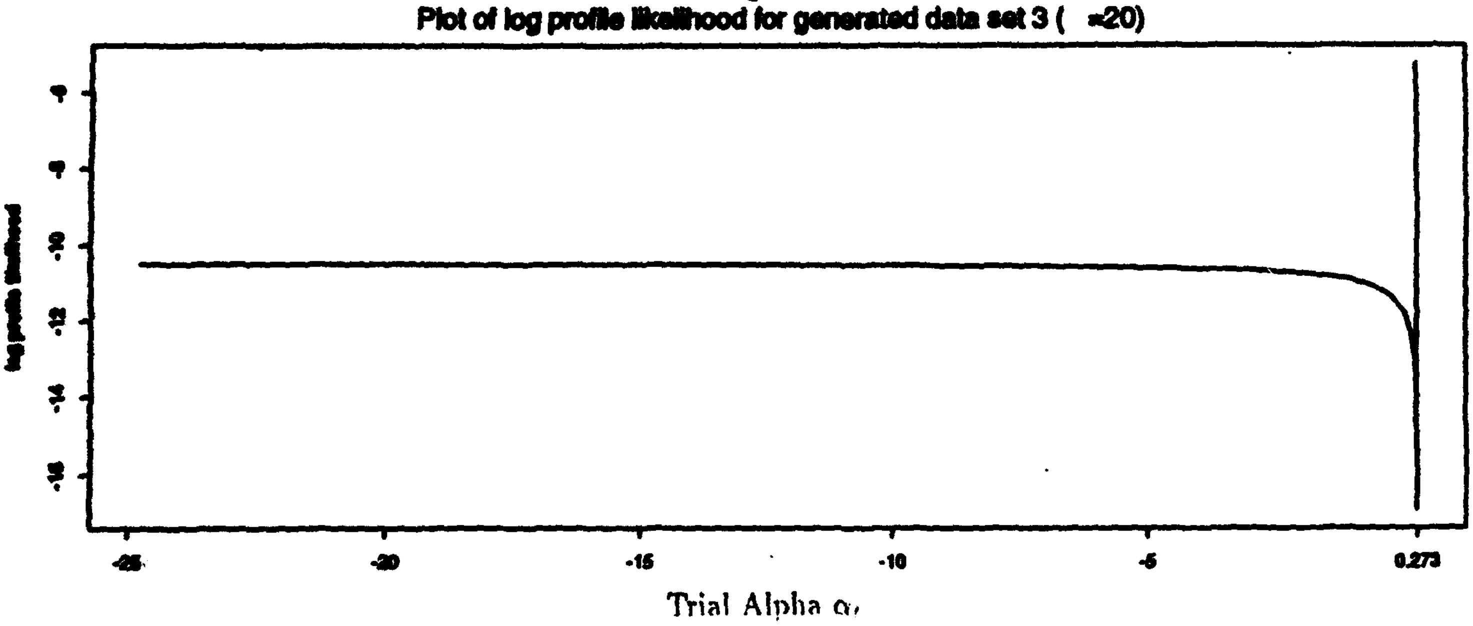
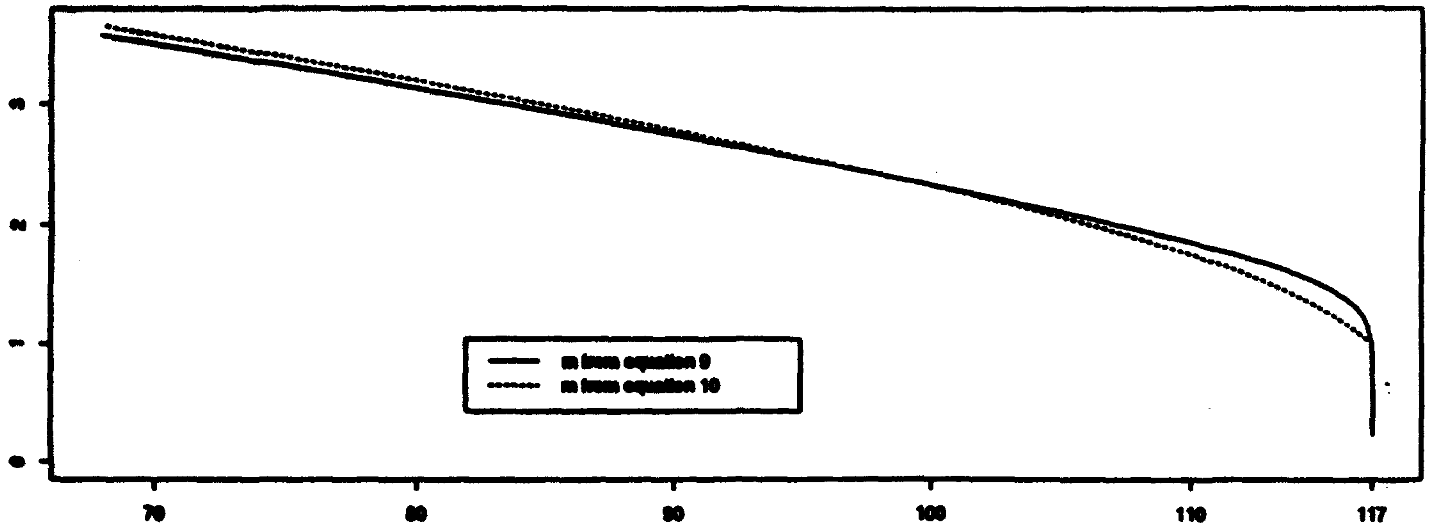
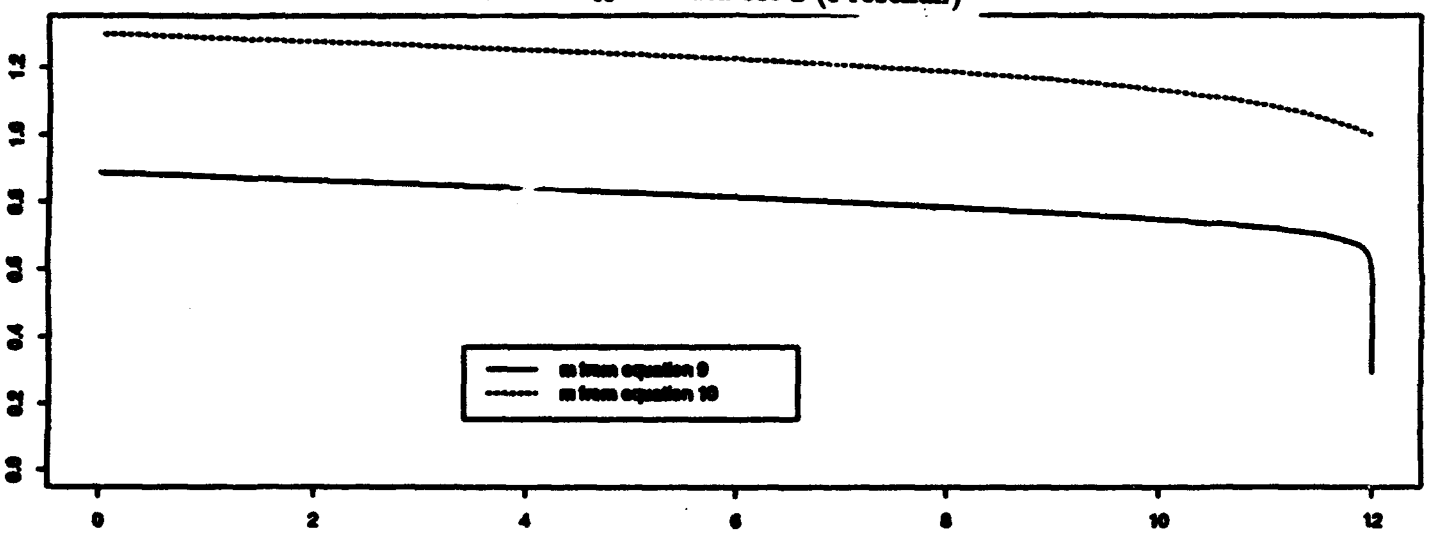


Figure 2a
m_9 and m_{10} for data set 1 (Cox and Oakes)



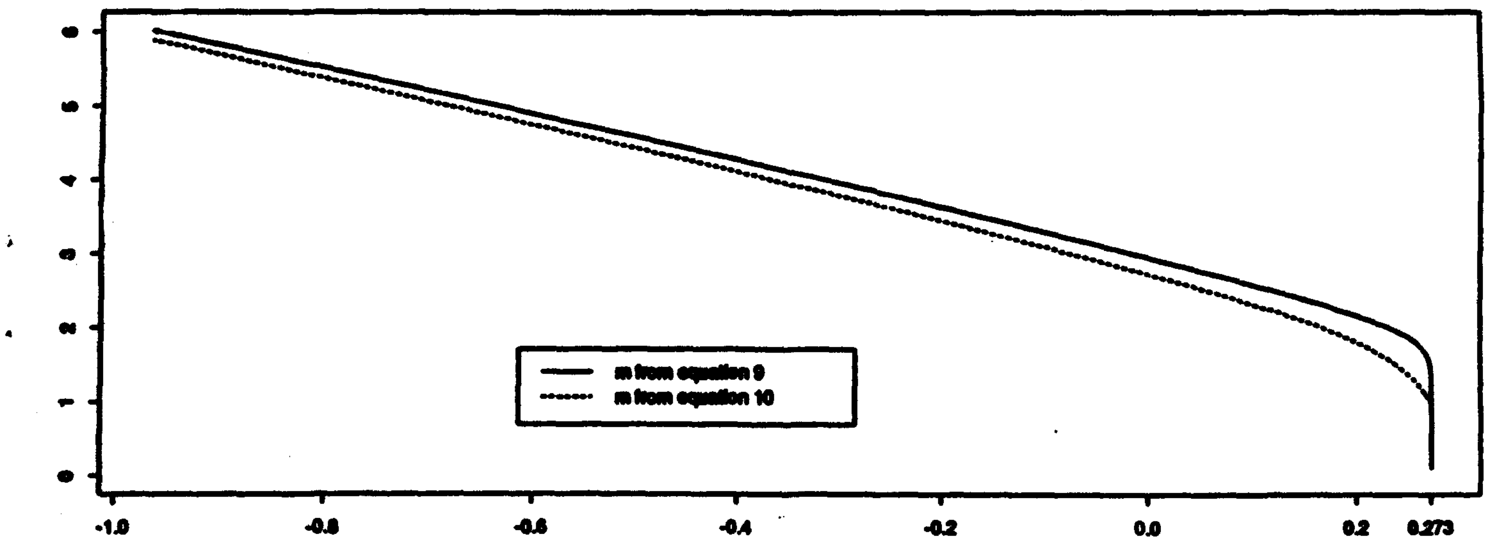
Trial Alpha α_t

Figure 2b
m_9 and m_{10} for data set 2 (Proschan)



Trial Alpha α_t

Figure 2c
m_9 and m_{10} for generated data set 3 ($n = 20$)



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Summary

Estimation techniques are given for the three-parameter Weibull distribution, with the location (or origin) parameter unknown, and possibly also the shape and/or scale parameters unknown. Tests of fit are described, and tables are given for the EDF statistics A^2, W^2 and U^2, to make the tests. Several examples are discussed.